Unique Monetary Equilibrium with Inflation in a Stationary Bewley-Aiyagari Model*

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Abstract

We prove the existence and uniqueness of a stationary monetary equilibrium in a Bewley-Aiyagari model with idiosyncratic shocks. This is an exchange economy with an infinite horizon and one consumption good, and with each agent facing idiosyncratic endowment shocks at each period; the agents may trade their endowments for the only asset, fiat money. The government increases the money supply at a constant growth rate that induces inflation in a stationary monetary equilibrium. We identify the necessary and sufficient condition for a stationary monetary equilibrium (where money has a positive value and the aggregate real balance is constant over time) to exist, and, when it exists, we show that it is unique. The argument for uniqueness is based on a new monotonicity result for the average optimal consumption.

JEL classification: E31, E40, E50.

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1 Introduction

The Bewley-Aiyagari model (Bewley [4], Aiyagari [3], and Hugget [9]) has become the standard environment to study heterogenous agents and endogenous distribution of asset holdings of various kinds. One use of such models is to study the welfare cost of inflation (see, for example, Imrohoroglu [11]), and this is typically done within the class of stationary equilibrium with a constant real balance. It has been demonstrated (Green and Zhou [7] and Wallace [24]) that the distributional aspect of monetary policy in these models is essential in determination of the optimal inflation rate. However, this endogenous distribution of money holdings also causes difficulty in deriving equilibrium existence and uniqueness. While equilibrium existence has been established, there are no general theoretical results for uniqueness of a stationary equilibrium in this type of model.[1]

In this paper we prove equilibrium existence and uniqueness in a pure currency economy under the Bewley-Aiyagari environment. In this model, there is a single perishable consumption good and a population of infinitely lived agents who face idiosyncratic endowment shocks at each period but may trade their endowments for money. The government increases the money supply at a constant growth rate in a lump-sum manner. We focus on stationary equilibria. In contrast to similar models in which the (rental) price of capital is determined by current aggregate capital holdings, in our economy the price of money also depends on its future prices and hence can only be determined endogenously. An equilibrium is *monetary* if money is valued in equilibrium. Stationarity requires two endogenous variables to be constant over time: first, the total real value of money (and, hence, when money supply grows at a positive rate, there is inflation); second, the distribution of money holdings. We give the necessary and sufficient condition for a monetary equilibrium to exist. More importantly, we show that when it exists, it is also unique.

In this environment, an agent's optimal money holding depends on his endowment

[1]As discussed in Miao [20] (comments after Definition 17.1.1), in the Bewley-Aiyagari model with capital accumulation, multiple equilibria cannot be excluded in general.

shock and his previous money holding, and hence follows a first-order Markov process. Under a constant money supply, equilibrium is determined by the unique invariant distribution of this Markov process. When money supply grows, however, the real value of the lump-sum transfer is also an equilibrium object, and market clearing gives an equilibrium condition that relates the average consumption under the equilibrium distribution and the real transfer. Uniqueness of equilibrium would follow if the average consumption is monotonic in the real transfer. The standard technique is to show that optimal consumption increases with the real transfer in the individual dynamic programming problem, which does not hold in general, however. Instead, we develop a new technique that directly demonstrates that average consumption increases with the real transfer. Our arguments rely on the ergodic theorem, which relates the long-run average of consumptions from an "typical" individual realization of endowment shocks to the average consumption of the stationary cross-section distribution of consumption. While similar existence results have been established in earlier papers, our uniqueness result is new.

Our uniqueness result implies real determinacy among stationary monetary equilibria. This is in contrast with the indeterminacy result in Green and Zhou [6], which finds a continuum of stationary monetary equilibria in the context of a Kiyotaki-Wright [13] model but with divisible money holdings and indivisible goods, and under double auctions. Moreover, since the unique equilibrium is upper hemi-continuous in inflation rate, the equilibrium *allocation* is also continuous. This implies that an optimal inflation rate exists, and the literature has provided examples in which such a rate is strictly positive, such as Green and Zhou [7].[2] In contrast with the Lagos-Wright [16] model, in which the unique stationary monetary equilibrium features a degenerate distribution and inflation through lump-sum transfers is never optimal, in our model a monetary equilibrium always features a non-degenerate distribution and inflation can be optimal.[3]

[2]Other examples in closely related models include Rocheteau, Weill, and Wong [21] based on variants of the Lagos-Wright [16] model, and Lippi, Ragni and Trachter [19] based on the Scheinkman-Weiss [22] model.

[3]See Gu and Wright [8] for a uniqueness result in the context of the Lagos-Wright model.

More broadly speaking, our paper contributes to a recent literature on the existence and uniqueness of equilibria in Bewley-Aiyagari models. Açikğoz [2] proves the existence of a stationary equilibrium in such a model with capital accumulation, but, in contrast to our pure-currency economy, he also demonstrates that there can be multiple stationary equilibria. As pointed out there, one reason for this multiplicity is the income effects of a higher rate-of-returns on assets, and Lehrer and Light [17] give a sufficient condition on the underlying utility function for the substitution effect to dominate the income effect; Light [18] uses that condition to obtain uniqueness.

Finally, our uniqueness result allows for unambiguous comparative statics, and one can apply existing comparative-statics results to our setup. In particular, although our setup is not a special case of that considered in Acemoglu and Jensen [1], we can readily translate their results in our setup. For example, given their results, it is easy to show that an increase in the discount factor will lead to an increase in the equilibrium real balances in our setup.

2 Model

2.1 Environment

Time is discrete and there is an infinite horizon. There is a population of agents who are ex ante identical and there is one perishable good. Let $u(c)$ be the agent's utility from consuming $c \geq 0$ units of the good. We assume that $u : \mathbf{R}_+ \to [0, \bar{u}]$, where $u(0) = 0$ is bounded, strictly increasing and strictly concave.[4] Agents maximize discounted expected utility with discount factor $\beta \in (0, 1)$. At the beginning of each period each agent receives an idiosyncratic shock to his endowment, denoted by y. We assume that y is drawn from a closed interval $Y = [\underline{y}, \overline{y}]$ with $0 \leq \underline{y} < \overline{y}$ and is i.i.d. across periods from distribution $\pi \in \Delta(Y)$ such that $\underline{y}, \overline{y} \in \text{support}(\pi)$. Agents cannot commit to future actions and there

[4] We assume the utility function to be bounded to avoid technical issue with the dynamic programming problem. See discussion in Section 4 for more details.

is no record-keeping technology for credit arrangements. There is an intrinsically useless asset called money, and in each period agents may trade their money holdings against the consumption good in a competitive market.

We assume that the government increases the money supply at a constant net growth rate $\gamma \geq 0$; thus, if M_t is the initial average money holding at period t, then

$$M_t = (1+\gamma)M_{t-1} \text{ for each } t = 1, 2, ...,$$

and this increase in money supply is achieved with a lump-sum transfer: in the end of each period $t = 0, 1, 2, ...$, each agent receives γM_t units of money from the government.

It is convenient to normalize the agent's money holdings as fractions of the average money holding. With this normalization, the above scheme of increasing money supply at rate $\gamma \geq 0$ with a lump-sum transfer is equivalent to the following scheme with a proportional taxation on money holdings and with a lump-sum transfer: an agent who holds m units of money before the transfer will end up with $(m+\gamma)/(1+\gamma) = (1-\tau)m+\tau$ units after the transfer, where $\tau = \gamma/(1+\gamma)$. We call τ the *tax rate on money holding*. Thus, the course of action in each period has three stages: first, agents receive their endowments; second, agents trade between their endowments and money; finally, the tax on money holding and transfer occurs. Note also that under this normalization the average money holding is one for all periods.

Fix a sequence $p_0, p_1, p_2, \dots \in \mathbf{R}_+$ of *prices of money in terms of goods*. Consider a single agent who maximizes expected discounted utility assuming the price of money in terms of goods at every period t is p_t. The agent's problem is a dynamic optimization problem with state variable (m, y) representing money holding and endowment before the trade. Let $V_t(m, y)$ be the optimal continuation value under state (m, y) at period t.

Compressing the dependence on $p_0, p_1, \ldots$, the Bellman Equation for V_t is given by

$$V_t(m,y) = \max\left\{u(c) + \beta \int V_{t+1}\left((1-\tau)m' + \tau, y'\right)\ \pi(\mathrm{d}y') : 0 \le c, m',\ c + m'p_t \le y + mp_t\right\} \tag{1}$$

for every $t \ge 0$. Here the choice variables c and m' are the consumption and post-trade money holding. We use $c_t(m,y)$ and $\phi_t(m,y)$ to denote the optimal consumption and post-trade monetary holding that correspond to the sequence $p_0, p_1, \ldots$ of prices.

Definition 1. For a given initial distribution of money holdings, $\mu_0 \in \Delta(\mathbf{R}_+)$, an *equilibrium with a tax rate on money holding* τ is a sequence $p_0, p_1, \ldots$ of prices of money in terms of goods, and a sequence $\mu_1, \mu_2, \dots \in \Delta(\mathbf{R}_+)$ of distributions of money holdings with $\int m\ \mu_t(\mathrm{d}m) = 1$ for all $t \ge 0$, such that the following holds for every $t \ge 0$:[5]

1. Law of motion: $\mu_{t+1}(A) = \mu_t \otimes \pi\left(\{(m,y) : (1-\tau)\phi_t(m,y) + \tau) \in A\}\right)$ for every Borel subset A of $\mathbf{R}$.

2. Market clearing: $\int c_t(m,y)\ \mu(\mathrm{d}m)\ \pi(\mathrm{d}y) = \int y\ \pi(\mathrm{d}y)$.

The equilibrium is *monetary* if $p_t > 0$ for every $t \ge 0$.

The law of motion reflects the fact that if the distribution of money holdings at the beginning of period t is μ_t and agents trade optimally then the distribution of money holdings at the beginning of period $t+1$ is given by μ_{t+1}. The market-clearing conditions express the clearing of the market for the consumption good at each period. By Warlas' Law there is a sequence of equivalent conditions in terms of money holdings.

It is easy to verify that there is a nonmonetary equilibrium under which $p_t = 0$ for all $t \ge 0$. In this equilibrium money has no value and all agents consume their endowments at each period. In contrast, we are interested in a monetary equilibrium under which money has a positive value.

[5]We use $\mu_t \otimes \pi$ to denote the joint distribution of money holdings and endowments under independence of the two.

While our setup is closely related to that in Acemoglu and Jensen [1], there are two differences. First, in their setup each agent solves a dynamic programming problem that depends on some parameter (that corresponds to p_t in our model), which, under market clearing, is determined by an exogenous function of the aggregation of individual choices. In contrast, the sequence $\{p_t\}_{t=0}^{\infty}$ in our setup is endogenous and the market clearing conditions relate the aggregation of individual choice to the aggregation of the endowments. Second, our interest in monetary equilibrium has no counterpart in their environment. For these reasons we cannot directly use their results in our setup. In particular, the question for which initial distribution of money holdings μ_0 there exists some monetary equilibrium is an open question in our setup.

Remark 1. The conditions in Definitions 1 can be interpreted in two ways: from the perspective of a single agent, and from the perspective of the population. Consider first a single agent whose initial money holding is randomized from μ_0, who receives a stochastic stream of shocks, and who consumes and saves according to c_t and ϕ_t. The time series of consumption and money holding of the agent is then a stochastic process. The market-clearing conditions imply that μ_t is the distribution of money holding at day t and that the expectation of period t's consumption equals the expected endowment. At the population level, the dynamic is completely deterministic: μ_t is the empirical distribution of money holdings at the beginning of period t and $\mu_t \otimes \pi$ is the empirical joint distribution of money holdings and endowments. More explicitly, we can identify the set of agent with $\mathbf{R}_+ \times [\underline{y}, \overline{y}]^{\mathbb{N}}$ equipped with a measure $\mu_0 \otimes \pi^{\otimes \mathbb{N}}$, so that each agent is identified with his initial money holding and the infinite sequence of endowments. From this population perspective, the market-clearing conditions say that the population consumes the total endowment. This dual perspective is a basic feature of the Bewley-Aiyagari models. We use the stochastic single-agent perspective to derive the optimal consumption and saving strategy of the agents, and we use the deterministic population perspective to describe what will actually happen.[6]

We focus on the stationary monetary equilibrium defined below.

Definition 2. An equilibrium with a tax rate on money holding τ is *stationary* if $p = p_0 = p_1 = \ldots$ and $\mu = \mu_0 = \mu_1 = \ldots$ for some $p \in \mathbf{R}_+$ and $\mu \in \Delta(\mathbf{R}_+)$ with $\int m \ \mu(\mathrm{d}m) = 1$. The stationary equilibrium is monetary if $p > 0$.

In a stationary monetary equilibrium with tax rate τ, the price for money is constant over time. However, remember that this is because we normalized the average money supply to one unit per agent, and hence, the price for money before the normalization (the version where the money supply increases at rate γ per period) decreases at a constant rate. This then implies that there is a constant inflation at rate γ and the (gross) rate-of-return on money is $1 - \tau$.

Note that under a stationary equilibrium the stochastic process that represents the consumption and saving of a single agent (as mentioned in Remark 1) is stationary. By the ergodic theorem, the expectation of consumption at each period also equals the average long-run realized consumption. Therefore, under a stationary equilibrium, the realized (across-period) average consumption of the agent equals the mean endowment.

Now we are ready to present our main result.

Theorem 1. *There exists a stationary monetary equilibrium if and only if*

$$u'(\overline{y}) < \beta(1-\tau) \cdot \int u'(y) \ \pi(\mathrm{d}y). \tag{2}$$

When it exists, it is also unique and the invariant distribution μ of money holding is non-degenerated.

Theorem 1 gives a precise condition for a monetary equilibrium to exist, (2), and when it does, it shows that it is also unique. This condition essentially gives an upper bound

[6] Readers who would like us to embed the two perspectives in a single model, with stochastic processes for each agent and appeal to some "exact law of large numbers" in order to justify "no aggregate uncertainty" are referred to Section II.B in Acemoglu and Jensen [1] and the references therein.

on the inflation rate (note that $1/(1-\tau) = 1+\gamma$). When the utility function u satisfies the Inada condition that $u'(0) = \infty$ (such as the typically used CRRA utility functions) and when π gives sufficiently high probability to arbitrary small endowments, this upper bound becomes infinite. Otherwise, this upper bound increases when the discount factor becomes larger and when π decreases in the standard stochastic order.

The condition (2) is derived from the following exercise: given that the (gross) rate of return on money is $1-\tau$ and that there is no government transfer, would an agent with the highest endowment, $\bar{y}$, and with no money at hand, save any positive amount of money? We use the Euler equation to show that the answer is yes if and only if condition (2) holds (see Claim 3 in the proof section). As a result, when the condition (2) fails, monotonicity of the saving function implies that no agent would save (see Claim 4 in the proof section), and hence the only stationary equilibrium is nonmonetary and the equilibrium allocation is autarky. In contrast, when the condition (2) holds, we show that a monetary equilibrium exists. This existence part of Theorem 1, though apparently new, follows from standard arguments.[7] Our main contribution is to show that such an equilibrium is unique.

Uniqueness allows for an unambiguous comparative statics, and one may directly apply some known results. In particular, although our setup is not a special case of that considered in Acemoglu and Jensen [1], it is easy to show that an increase in any "positive shocks" defined there (i.e., any changes in exogenous parameters that will lead to an increase in the policy function ϕ) will lead to an increase in the equilibrium real balances in our setup. One such shock is an increase in the discount factor β. However, an increase in the endowment (in the first-order-stochastic-dominance sense) is not a positive shock in general, as it has two opposing effects to the optimal money holding: while it gives more resources to save today, it also gives a better future so that there is less need

[7]Geanakoplos et al. [5] also give an existence result in a closely related economy with inflation. However, because a cash-in-advance constraint is assumed there, there is no need for any condition analogous to (2).

to save today (the income effect).

3 Proof of Theorem 1

After some mathematical preliminaries (Section 3.1), we first write the Bellman equation for the single-agent problem and the definition of stationary equilibrium in terms of real values (Proposition 3 in Section 3.2). With this formulation, the state space represents the real wealth of the agent at every period after the endowment shock.

The argument for proving existence is standard. Since, for each individual agent, the optimal real balances across periods follow a Markov process, we show that this process satisfies the mixing condition in Stokey and Lucas [23] and hence has a unique ergodic distribution. When $\tau = 0$, that ergodic distribution fully describes the unique stationary monetary equilibrium. When $\tau > 0$, however, a fixed-point argument is needed to establish the existence, because the single-agent problem depends on the real value of the monetary lump-sum transfer, denoted by $b = p\tau$, which in turn depends on the equilibrium price of money, p. To prove uniqueness of the fixed point when $\tau > 0$, we need some sort of monotonicity with respect to b. The core of our argument is that, while the individual policy function may not be monotonic, we can show that the average optimal consumption is increasing in b. We give this monotonicity result in Section 3.3, where we study the individual dynamic programming problem. Finally, we use the monotonicity result to prove Theorem 1 in Section 3.4.

3.1 Preliminaries

3.1.1 Notations

If $\mu, \nu \in \Delta(\mathbf{R})$ are probability distributions and $f : \mathbf{R} \to \mathbf{R}$, we denote by $f(\mu) \in \Delta(\mathbf{R})$ the push-forward of μ under f. This is the distribution of $f(X)$, where X is a random variable with distribution μ. In the special case that $f(x) = ax + b$, we also denote

$f(\mu) = a\mu + b$. We denote by $\mu * \nu$ the convolution of μ and ν. This is the distribution of $X + Y$, where X, Y are independent random variables with distributions X and Y.

Monotone markov chains

This section reviews known results about monotone Markov processes that will be used in the proof. Let $(W, \mathcal{W})$ be a Borel space. A *transition probability* over W is a function $\chi : W \times \mathcal{W} \to [0, 1]$ such that $\chi(w, \cdot)$ is a probability distribution over $(W, \mathcal{W})$ for every $w \in W$ and $\chi(\cdot, A)$ is a Borel function for every $A \in \mathcal{W}$. We let $T : \Delta(W) \to \Delta(W)$ be the *stochastic operator of* χ given by $T(\lambda)(A) = \int \chi(w, A)\lambda(\mathrm{d}w)$ for every $A \in \mathcal{W}$.

Fix a transition probability χ over $(W, \mathcal{W})$. For every $\lambda \in \Delta(W)$, we denote by P_λ the distribution of a sequence $X_0, X_1, \ldots$ of W-valued random variables such that

$$\begin{aligned} &X_0 \sim \lambda, \text{ and} \\ &P_\lambda(X_{k+1} \in \cdot | X_0, \ldots, X_k) = \chi(X_k, \cdot) \text{ for every } k \geq 0. \end{aligned} \tag{3}$$

When $\lambda = \delta_w$ is the Dirac measure over $w \in W$, we also denote $P_\lambda = P_w$. A probability distribution λ over W is called an *invariant distribution of* χ if $T\lambda = \lambda$. Equivalently λ is an invariant distribution if the stochastic process $X_0, X_1, \ldots$ given in (3) is stationary. For every $B \in \mathcal{W}$, let T_B be the (possibly infinite) first time in which the process hits B:

$$T_B = \inf\{1 \leq t : X_t \in B\}.$$

The invariant distribution λ is *ergodic* if $\mathbf{P}(T_B < \infty) = 1$ for every $B \in \mathcal{W}$ such that $\lambda(B) > 0$. The fundamental feature of ergodic distributions, captured by the Ergodic Theorem [20, Theorem 4.3.2], is that the expected value of a function $f : W \to \mathbf{R}$ of the state of the process at any given time almost surely equals the time average of the function of the sample path of the process.

For our argument we use the following lemma (Kac's Lemma, c.f. Krengel [14], Proposition 6.8): The expectation of a function f of the state in any given time equals the expected average of the function between two entries to a set B. The lemma follows immediately from the ergodic theorem, but in fact it follows from more basic principles.

Lemma 1. *Let χ be a transition probability over $(W, \mathcal{W})$, and let λ be an ergodic invariant distribution. Let $f : W \to \mathbf{R}$ be Borel measurable and bounded. Then for every $B \in \mathcal{W}$ such that $\lambda(B) > 0$, it holds that*

$$\mathbb{E}f(X_0) = \mathbb{E}\left(1_{\{X_0 \in B\}} \sum_{t=0}^{T_B - 1} f(X_t)\right),$$

where $X_0, X_1, \ldots$ are the W-valued random variables given by (3).

The transition probability χ is *uniquely ergodic* if it admits a unique invariant distribution λ. If χ is a uniquely ergodic transition probability, then its invariant distribution is ergodic.

We assume from now on that $W = [\underline{w}, \overline{w}] \subseteq \mathbf{R}^n$ is an n-dimensional interval, equipped with the standard compact lattice structure. The transition probability χ is *monotone* if $\chi(w, \cdot) \leq_{st} \chi(w', \cdot)$ whenever $w \leq w'$. Equivalently, χ is monotone if the corresponding stochastic operator $T : \Delta(W) \to \Delta(W)$ is monotone in first-order stochastic dominance. We present two propositions, Propositions 1 and 2, which are taken from Stokey and Lucas [23].[8]

Proposition 1. *Every monotone transition probability admits an invariant distribution.*

Proposition 2. *Let χ be a monotone transition probability. If there exists some $w \in \mathbf{R}^n$ such that $\underline{w} < w < \overline{w}$ and such that*

$$P_{\overline{w}}\left(X_k < w \text{ for some } k \geq 0\right) > 0, \text{ and } P_{\underline{w}}\left(X_k > w \text{ for some } k \geq 0\right) > 0, \tag{4}$$

[8]The existence result in Stokey and Lucas [23] for Proposition 1 requires Feller property, which, as noted by Hopenhayn and Prescott [10], is not necessary. The assertion about the support in Proposition 2 is not stated in Stokey and Lucas [23], but it follows immediately from the proof.

then χ is uniquely ergodic. Moreover, if (4) *holds for every such w, then $\underline{w}, \overline{w}$ are in the support of the invariant distribution of χ.*

3.2 Stationary equilibrium in real terms

The definition of stationary equilibrium (Definition 2) is based on a parametrized single agent problem, where the state (m, y) is money holding and current endowment and the price of money p is a parameter. For our proof, it is convenient to rewrite the agent's problem in real terms, with state space $\mathbf{R}_+$ representing the *real wealth* $w = pm + y$ of the agent at every period (after the endowment shock and before the trade), and parameter space $\mathbf{R}_+$ representing *real government transfer* b. The parameter b would correspond to the lump-sum transfer component, τ, in the original Bellman equation (1), but here it represents the *real value* of such transfer and its equilibrium value depends on the equilibrium value of money.

Let $V_b(w)$ be the corresponding value function, and the Bellman equation for $V_b(w)$ becomes

$$V_b(w) = \sup_{0 \le c \le w} \left\{ u(c) + \beta \int_{y \in Y} V_b\big((1-\tau)(w-c) + y + b\big)\ \pi(\mathrm{d}y) \right\}. \tag{5}$$

By standard dynamic programming arguments it follows from the assumptions on u that the supremum is achieved at a unique consumption level. We denote by $c_b(w)$ the optimal consumption and $s_b(w) = w - c_b(w)$ the optimal real balance holding at the end of a period.

Consider a stationary monetary equilibrium (μ, p) with distribution of money holding μ and price of money p. Under this equilibrium the real government transfer b and the distribution of real wealth λ are given by

$$b = \tau p, \quad \lambda = p\mu * \pi. \tag{6}$$

The following proposition characterizes stationary equilibrium in terms of the two real objects, b and λ.

Proposition 3. *For a fixed effective tax rate on money τ, a pair (μ, p) constitutes a stationary monetary equilibrium with τ if and only if there exist a distribution $\lambda \in \Delta(\mathbf{R}_+)$ and a number $b \geq 0$ for which (6) holds and which satisfy the following conditions:*

1. *Invariance: $\lambda = (1-\tau)s_b(\lambda) * \pi + b$.*

2. *Government balance: $b = \tau \int s_b \, \mathrm{d}\lambda$.*

3. *Positive real balances: $\int s_b \, \mathrm{d}\lambda > 0$.*

The first condition corresponds to the invariance condition in Definition 2, and it says that the distribution of real wealth is constant. In the second condition, the left side is the real value of the lump-sum monetary transfer to agents, and the right side is the inflation tax (in real terms) the government collects through money creation. The last condition corresponds to the requirement for a monetary equilibrium in Definition 1, that is, $p > 0$.

The conditions in Proposition 3 are written in a way that highlights the additional difficulty that is involved in proving uniqueness when $\tau > 0$. Indeed, for $\tau = 0$ the government-balance condition implies that $b = 0$. In this case, as we shall see, the existence and uniqueness of a stationary monetary equilibrium follows immediately from the existence and uniqueness of the invariant distribution of the Markov transition induced by the random endowment and the agent's optimal saving. For $\tau > 0$ we need to find the pair b and λ that satisfies the invariance condition and the government-balance condition simultaneously. This requires an appeal to some fixed-point argument (which is easy, because b is a one-dimensional entity). Uniqueness requires some monotonicity result, which is the main theoretical contribution of this paper.

Proof of Proposition 3. Suppose that (λ, b) satisfies the three listed conditions. Let

$$p = \int s_b \, \mathrm{d}\lambda \text{ and } \mu = \left((1-\tau)s_b(\lambda) + b\right)/p. \tag{7}$$

First we show that (λ, b) satisfies (6) for (p, μ). By invariance of λ (condition 1 in Proposition 3) and by (7),

$$\lambda = (1-\tau)s_b(\lambda) * \pi + b = p\mu * \pi.$$

Moreover, there is also an one-for-one translation between the policy functions induced by (1) and by (5): $p\phi(m, y) = s_b(pm + y)$. Thus, invariance of λ also implies invariance of μ:

$$p\mu * \pi = \lambda \text{ and hence } s_b(p\mu * \pi) = s_b(\lambda).$$

The other direction is immediate. □

3.3 The individual consumption-saving problem

In this subsection we study the single agent's problem (5). By standard dynamic programming arguments, it follows from the assumptions on u that V_b is bounded, continuous, monotone increasing, and strictly concave in w, and that V_b is submodular in (b, w); that the supremum is achieved at a unique consumption level and that the optimal consumption $c_b(w)$ is continuous, increasing in b, strictly increasing in w, and $\lim_{w\to\infty} c_b(w) = \infty$; that the optimal saving $s_b(w) = w - c_b(w)$ is decreasing in b and strictly increasing in w; that *Euler's equation*

$$u'(c_b(w)) \geq \beta(1-\tau) \int u'(c_b[(1-\tau)s_b(w) + y + b])\ \pi(\mathrm{d}y) \tag{8}$$

is satisfied, with equality if $c_b(w) < w$; and that $c_b(\cdot)$ is the unique function that satisfies Euler's equation (8) and the transversality condition.

Let

$$\eta_b(w, y) = (1-\tau)s_b(w) + y + b \tag{9}$$

be the next-period wealth of an agent who has wealth w at the current period, and gets the lump-sum transfer b and endowment shock y next period. The following claim summarizes properties of the next-period wealth function.

Claim 1. *Let $\eta_b(w, y)$ be the next period wealth function given by* (9). *Then $\eta_b(w, y)$ is continuous in b, w, y, monotone increasing in w and y, and $\eta_b(w, y) - w$ is strictly decreasing in w and increasing in y. Moreover, $\eta_b(w, \bar{y}) - w < 0$ for a sufficiently large w.*

Proof. The first assertion follows from (9) and the corresponding properties of the optimal saving function. The second assertion follows from $\eta_b(w, y) - w = -c_b(w) - \tau s_b(w) + y + b$ and monotonicity of the optimal consumption and saving functions. The last assertion follows from the fact that $\lim_{w \to \infty} c_b(w) = \infty$. □

Invariant distribution

By Claim 1, there exists a unique $\overline{w}_b \in \mathbf{R}_+$ such that $\eta_b(\overline{w}_b, \bar{y}) = \overline{w}_b$; let $\underline{w}_b = b + \underline{y}$ so that $\eta_b(\underline{w}_b, \underline{y}) = \underline{w}_b$ since $c_b(b + \underline{y}) = b + \underline{y}$ from the Euler's equation. Consider the transition probability χ_b on $W = [\underline{w}_b, \overline{w}_b]$ such that $\chi_b(\cdot|w) = \eta_b(w, \pi)$. The corresponding stochastic operator is given by

$$T_b(\lambda) = (1 - \tau)s_b(\lambda) * \pi + b. \tag{10}$$

Then it follows from Claim 1 and the definition of $\underline{w}_b, \overline{w}_b$ that χ_b is a monotone transition on $[\underline{w}_b, \overline{w}_b]$. The following claim and its proof are standard.

Claim 2. *For every $b \geq 0$, the transition χ_b is uniquely ergodic. Moreover, if λ_b is the invariant distribution then $\underline{w}_b, \overline{w}_b \in support(\lambda_b)$.*

Proof. Using Proposition 2 we need to show that (4) holds for every w such that $\underline{w}_b < w < \overline{w}_b$. Indeed, let $\underline{y} < y$ be sufficiently small such that $\eta_b(w, y) < w$, with existence following from the definition of $\underline{w}_b$ and Claim 1. Then it follows from Claim 1 that $\eta_b(w', z) < w$ whenever $w' < w$ and $\underline{y} \leq z \leq y$, and that $\eta_b(w', z) < w'$ whenever $w' \geq w$

and $\underline{y} \leq z \leq y$. These properties and the continuity of η imply that there exists some N such that, for every sequence, $z_1, \ldots, z_N$, of endowments such that $z_i \in [\underline{y}, y]$, receiving these endowments in consecutive periods will decrease the agent's wealth to below w, regardless of the initial wealth. Since there is a positive probability that the endowment of the agent at a given period is in $[\underline{y}, y]$, the first condition of Proposition 2 is indeed satisfied. The second condition is proved analogously. $\square$

Monotonicity

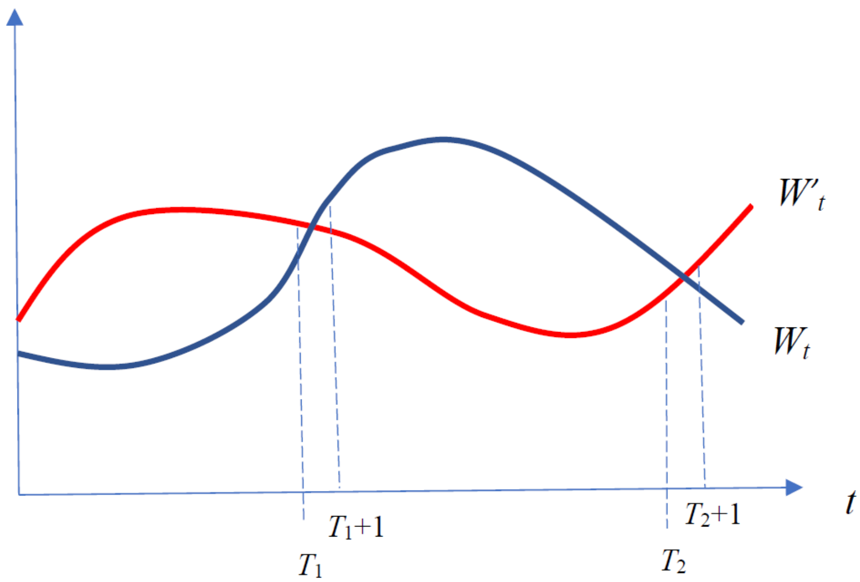
Although in general the function $b \mapsto (1-\tau)s_b(w)+b$ is not monotonic in b and hence the transition χ_b is not monotonic in b (w.r.t. first-order-stochastic-dominance), the following lemma, which is the core of our proof, says that the *average* consumption is indeed monotonic in b.

Lemma 2. *Let λ_b be the unique ergodic distribution given by Claim 2. The function $b \mapsto \int c_b \, \mathrm{d}\lambda_b$ is continuous and strictly increasing in b.*

Note that by the Ergodic Theorem $\int c_b \, \mathrm{d}\lambda_b$ is the long-run average of individual consumption. Lemma 2 is very intuitive: the average consumption of an agent increases if the agent receives a larger transfer every period. However, the lemma is not obvious since the agent's strategy maximizes discounted utility instead of average consumption. Standard arguments for proving comparative statics on the invariant distribution of a Markov transition with respect to a change in a parameter require that the transition be monotonic in the parameter. Such changes are called positive shocks in Acemoglu and Jensen [1] and underline all the comparative-statics results in their paper and, to our knowledge, all other known similar results. In our setup, however, increasing b is not a positive shock—the problem is that $\chi_b(w)$ is not increasing in b in sense of first-order stochastic dominance.

Here is a rough intuition for the proof: Consider two agents, one facing a transfer b and another b' with $b < b'$. We call the former agent Low and the latter High. We couple

Figure 1: **Coupled wealth process**



the stochastic processes of consumption and saving of High and Low by assuming they receive the same daily endowment. Consider a typical realization of wealth for these two agents, as depicted in Figure 1. Each period in which High starts with higher wealth than Low, it follows from properties of the optimal consumption that High consumes more than that of Low, as would be the case for the periods within period 0 to period T_1 in the above figure. Consider the interval from period $T_1 + 1$ to period T_2, an interval of consecutive periods at the beginning of which High's wealth is lower than Low. Then during these periods High paid less taxes than Low, and received the same endowment and higher transfer in each period. Moreover, if we count the interval from period T_1, then High entered this sequence with more wealth than Low and finished it with less wealth. It then follows that in all these periods taken together High consumed more than Low. Therefore, in the long-run High consumes more than Low on every realization. By the Ergodic Theorem, the long run average consumption on each realization equals the expected consumption of the agents at each period. Therefore, the expected consumption of High is higher than that of Low. This is the basic idea of the proof with two caveats:

first, we need to be more careful with what happens in the endpoints of the sequence of days in which High has lower wealth than Low; second, the ergodic theorem delivers the right intuition but it is overkill. In fact, we only need the more basic Kac's Lemma.

Proof of Lemma 2. Continuity of the next period wealth function $(b, w, y) \mapsto \eta_b$ in b, w implies that the function $(b, w) \mapsto \chi_b(\cdot|w)$ is continuous in the weak* topology. By Theorem 12.13 in Stokey and Lucas [23], this implies continuity of the map $b \mapsto \lambda_b$. Because the map $(b, w) \mapsto c_b(w)$ is also continuous it follows that the map $b \mapsto \int c_b \, \mathrm{d}\lambda_b$ is continuous.

Let $b < b'$, and consider a coupling,

$$(W_0, W_0', Y_0), (W_1, W_1', Y_1), \ldots, (W_t, W_t', Y_t), \ldots,$$

of the wealth process of the two agents with the same endowment process Y_t, where one agent, called Low, receives transfer b at every period, and a second, called High, receives transfer b'. The wealth of the agents at the beginning of period t is W_t and W_t'. Therefore $W_t = \eta_b(W_{t-1}, Y_{t-1})$ and $W_t' = \eta_{b'}(W_{t-1}', Y_{t-1})$. By the same argument as in Claim 2 the process is ergodic.[9] We have to prove that $\mathbb{E}\{c_{b'}(W_0')\} > \mathbb{E}\{c_b(W_0)\}$. By Lemma 1, we need to prove that

$$\mathbb{E}\left(1_{\{W_0 < W_0'\}} \sum_{t=0}^{T-1} (c_{b'}(W_t') - c_b(W_t))\right) > 0,$$

where $T = \inf\{1 \le t < \infty : W_t < W_t'\}$. Note that the event $W_0 < W_0'$ is indeed of positive probability, which follows from the assertion about the support of λ_b and $\lambda_{b'}$ in Claim 2 and the fact that $\underline{w}_b < \underline{w}_{b'}$. We prove the stronger assertion that

$$\sum_{t=0}^{T-1} (c_{b'}(W_t') - c_b(W_t)) > 0 \text{ a.s.} \tag{11}$$

on the event $\{W_0 < W_0'\}$. Indeed, if $T = 1$, then the inequality follows from the fact that

[9] One has to apply the argument in Claim 2 to show that the probability transition over $[\underline{w}_b, \overline{w}_b] \times [\underline{w}_{b'}, \overline{w}_{b'}]$ is uniquely ergodic.

$W_0 < W_0'$ and the monotonicity of $c_b(w)$ (both in w and in b). Assume now that $T > 1$. Let $0 < t < T$. Then $W_t \geq W_t'$ and therefore

$$s_b(W_t) > s_{b'}(W_t') \tag{12}$$

from the monotonicity of $s_b(w)$ (increasing in w and decreasing in b). In addition, (12) holds for $t = 0$ since $W_1 = (1-\tau)s_b(W_0) + b$ and $W_1' = (1-\tau)s_{b'}(W_0') + b'$, but $b < b'$ and $W_1 \geq W_1'$ since $T > 1$. Thus,

$$\begin{aligned}
\sum_{t=0}^{T-1} c_b(W_t) &= W_0 - \tau \sum_{t=0}^{T-2} s_b(W_t) + \sum_{t=0}^{T-2} Y_t + (T-1)b - s_b(W_{T-1}) \\
&< W_0' - \tau \sum_{t=0}^{T-2} s_{b'}(W_t') + \sum_{t=0}^{T-2} Y_t + (T-1)b' - s_{b'}(W_{T-1}') = \sum_{t=0}^{T-1} c_{b'}(W_t'),
\end{aligned}$$

where the equalities follow from the aggregation of the households' consumption and transfers from the beginning of period 0 until the market on period $T-1$ closes, and the inequality follows from (12) and the fact that $W_0 < W_0'$, $b < b'$. This proves (11). $\square$

Remark 2. The same proof can be used to prove a more general property in our setting: the long-run average consumption in increases when the endowment increases in first-order stochastic dominance. The only change we need to make in the proof is to couple endowment shocks of the two agents so that the high agent receives a higher endowment every period.

3.4 Proof of Theorem 1

The following claims state the implications of condition (2) in Theorem 1 in terms of the individual agent's problem: Claim 3 implies that, when the lump-sum transfer is zero, the agent will save some money after receiving a high endowment, and Claim 4 implies that in this case the agent will save money under the invariant distribution.

Claim 3. *Condition* (2) *holds if and only if* $s_0(\overline{y}) > 0$.

Proof. If (2) holds, then it follows from Euler's equation that the optimal consumption c_0 satisfies $c_0(\overline{y}) < \overline{y}$ so that $s_0(\overline{y}) = \overline{y} - c_0(\overline{y}) > 0$. If (2) does not hold, then the function $\tilde{c}_0(w) = w$ for every $w \in [\underline{y}, \overline{y}]$ satisfies Euler's equation and the transversality condition and therefore it is optimal. Therefore, the optimal s_0 satisfies $s_0(\overline{y}) = \overline{y} - c_0(\overline{y}) = 0$ □

Claim 4. *Condition* (2) *holds if and only if*

$$\int s_0 \, \mathrm{d}\lambda_0 > 0. \tag{13}$$

Proof. Note that $\int s_0 \, \mathrm{d}\lambda_0 > 0$ if and only if $s_0(\overline{w}_0) > 0$, because $\overline{w}_0$ is the maximal element in support(λ_0) and $s_0(\cdot)$ is monotonic and continuous.

We use Claim 3. If (2) holds, then $s_0(\overline{y}) > 0$, and therefore $\eta_0(\overline{y}, \overline{y}) > \overline{y}$ which implies $\overline{w}_0 > \overline{y}$ by the definition of $\overline{w}_0$ and Claim 1. Therefore, $s_0(\overline{w}_0) \geq s_0(\overline{y}) > 0$ by the monotonicity of $s_0(w)$. If (2) does not hold, then $s_0(\overline{y}) = 0$, and therefore $\eta_0(\overline{y}, \overline{y}) = \overline{y}$, which implies $\overline{w}_0 = \overline{y}$ by definition of $\overline{w}_0$. Therefore, in this case $s_0(\overline{w}_0) = s_0(\overline{y}) = 0$. □

We are now ready to complete the proof of Theorem 1. It follows from the argument in Section 3.3 that λ_b is the unique distribution that satisfies Condition 1 (invariance) in Proposition 3 for every $b \geq 0$.

We claim that there exists a unique $b \geq 0$ such that Condition 2 (government balance) is satisfied. We consider two cases. First, suppose that $\tau = 0$. Then, clearly $b = 0$ is such a unique b. Suppose then that $\tau > 0$. From the invariance of λ_b, we get that

$$\int w \; \lambda_b(\mathrm{d}w) = (1 - \tau) \cdot \int s_b \, \mathrm{d}\lambda_b + b + \int y \; \pi(\mathrm{d}y).$$

Since $\int w \; \lambda_b(\mathrm{d}w) = \int s_b \, \mathrm{d}\lambda_b + \int c_b \, \mathrm{d}\lambda_b$ (which follows from $w = c_b(w) + s_b(w)$), we get that

$$b - \tau \cdot \int s_b \, \mathrm{d}\lambda_b = \int c_b \, \mathrm{d}\lambda_b - \int y \; \pi(\mathrm{d}y)$$

for every $b \geq 0$. Thus, it is sufficient to show that there exists a unique b for which the right-hand side of the above equation equals zero. The assertion about uniqueness now follows from Lemma 2 about strict monotonicity of $\int c_b \, \mathrm{d}\lambda_b$. The existence follows form the fact that the left-hand side of the equation is non-positive at $b = 0$ (and is strictly negative at $b = 0$ iff (2) holds by Claim 4), and that $\int c_b \, \mathrm{d}\lambda_b \xrightarrow[b \to \infty]{} \infty$, since $c_b(\underline{w}_b) \geq \underline{w}_b \geq b$ for every $w \geq b$.

Finally, we need to prove that Condition 3 in Proposition 3 (positive saving) holds if and only if (2) holds. Recall that by Claim 4, condition (2) is equivalent to (13). Again we consider two cases. If $\tau = 0$, then $b = 0$ and so a positive saving holds iff (13) holds. Suppose now that $\tau > 0$. To satisfy the government budget balancedness, both (13) and a positive saving are equivalent to $b > 0$, and the result follows immediately from Claim 4. This completes the proof of Theorem 1. Note that we only need the machinery of Lemma 2 when $\tau > 0$.

4 Concluding Remarks

The main contribution of this paper is to prove uniqueness using the coupling argument: comparing the mean value of two stationary process by embedding them . In the context of wealth processes of two agents under consumption-saving problems with different parameters, the coupling is done by assuming the two agents get the same endowment shocks in every period. We used this argument to establish the monotonicity result in Lemma 2, and applied this result to establish uniqueness of equilibrium in a pure-currency economics. In order to focus on the coupling argument, we have simplified all other components in the model. Below we discuss our assumptions, and highlight ones that are technical in natural but can be relaxed.

Unbounded utilities and endowments

We assume a bounded utility function to avoid technical issues with the individual dynamic programming problem (see Kuhn [15] and references therein for issues regarding unbounded utility functions). However, our argument for the uniqueness of the stationary monetary equilibrium does not depend on that, and our results will go through as long as the Bellman equation has a unique solution. Similarly, we can relax the assumption that the endowment is drawn from a compact set, but then we need to make distributional assumptions as in Kamihigashi and Stachurski [12] to ensure the existence and uniqueness of the invariant distribution for each b.

Markovian endowment process

For our coupling argument to work, the endowment process needs not to be i.i.d., and a Markovian endowment process or more generally any stationary process would work just fine, as long as the induced wealth process is uniquely ergodic (see Hopenhayn and Prescott [10] for sufficient conditions). Under this condition, in order to get existence of monetary equilibrium we need to modify the condition (2) corresponding to the Euler equation with respect to the given Makovian process of endowments.

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